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Onset of superradiant instabilities in the composed Kerr-black-hole–mirror bomb

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ABSTRACT

It was first pointed out by Press and Teukolsky that a system composed of a spinning Kerr black hole surrounded by a reflecting mirror may develop instabilities. The physical mechanism responsible for the development of these exponentially growing instabilities is the superradiant amplification of bosonic fields confined between the black hole and the mirror. A remarkable feature of this composed black-hole–mirror-field system is the existence of a critical mirror radius, $r_{\text{m}}^{\text{stat}}$, which supports *stationary* (marginally-stable) field configurations. This critical ('stationary') mirror radius marks the boundary between stable and unstable black-hole–mirror-field configurations: composed systems whose confining mirror is situated in the region $r_{\text{m}} < r_{\text{m}}^{\text{stat}}$ are stable (that is, all modes of the confined field decay in time), whereas composed systems whose confining mirror is situated in the region $r_{\text{m}} > r_{\text{m}}^{\text{stat}}$ are unstable (that is, there are confined field modes which grow exponentially over time). In the present paper we explore this critical (marginally-stable) boundary between stable and explosive black-hole–mirror-field configurations. It is shown that the innermost (*smallest*) radius of the confining mirror which allows the extraction of rotational energy from a spinning Kerr black hole approaches the black-hole horizon radius in the extremal limit of rapidly-rotating black holes. We find, in particular, that this critical mirror radius (which marks the onset of superradiant instabilities in the composed system) scales linearly with the black-hole temperature.

1. Introduction

One of the most intriguing phenomena in black-hole physics is the superradiant scattering of bosonic fields by spinning black holes: it was first pointed out by Zel'dovich [1] that a co-rotating integer-spin wave field of frequency ω interacting with a spinning Kerr black hole (BH) can be amplified (gain energy) if the composed system is in the superradiant regime [2]

$$\omega < \omega_{\text{c}} \equiv m\Omega_{\text{H}}. \tag{1}$$

Here m is the azimuthal harmonic index of the incident wave field, and

$$\Omega_{\text{H}} = \frac{a}{r_+^2 + a^2} \tag{2}$$

is the angular velocity of the spinning Kerr BH [3]. The amplification of the scattered bosonic fields in the superradiant regime (1) is accompanied by a decrease in the rotational energy and angular momentum of the central spinning BH.

Soon after Zel'dovich's discovery [1] of the superradiance phenomenon in BH physics, it was realized by Press and Teukolsky [4] that the coupled BH–field system may develop exponentially growing instabilities. This unstable system, known as the *black-hole bomb*, is composed of three ingredients [4]: (1) a spinning BH whose rotational energy serves as the energy source of the composed system, (2) a co-rotating bosonic cloud which orbits the central BH and interacts with it to extract its rotational energy, and (3) a reflecting mirror which surrounds the central BH and prevents the amplified bosonic field from radiating its energy to infinity.

A remarkable feature of this composed physical system is the existence, for each given set $(\bar{a}, l, m)$ of the BH and field parameters [5], of a critical (*minimum*) mirror radius which marks the boundary between stable and unstable BH–mirror-field configurations. The critical ('stationary' [6]) mirror radius, $r_{\text{m}}^{\text{stat}}(\bar{a}, l, m)$, corresponds to *stationary* confined field configurations which are characterized by the critical superradiant frequency (1).

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The composed BH–mirror-scalar-field system was studied in the important work by Cardoso et al. [7–9]. In particular, it was shown in [7] that, for a given set of parameters $(\bar{a}, l, m)$, BH–mirror-field systems whose mirror radii lie in the regime $r_{\rm m} < r^{\rm stat}_{\rm m}(\bar{a}, l, m)$ are stable (the confined scalar mode decays in time) whereas BH–mirror-field systems whose mirror radii lie in the regime $r_{\rm m} > r^{\rm stat}_{\rm m}(\bar{a}, l, m)$ are unstable (the confined scalar mode grows exponentially over time).

The numerical results presented in [7] indicate that the critical ('stationary' [6]) mirror radius, $r^{\rm stat}_{\rm m}(\bar{a}, l, m)$, has the following three important features:

- For a confined field mode of given harmonic indexes (l, m), the stationary mirror radius is a decreasing function of the BH angular momentum $\bar{a}$. In other words, rapidly-rotating Kerr BHs are characterized by stationary mirror radii which are smaller (closer to the BH horizon) than the corresponding stationary mirror radii of slowly-rotating BHs.
- For a given value of the spheroidal harmonic index l of the confined field, the stationary mirror radius decreases with increasing values of the azimuthal harmonic index m. The equatorial $m = l$ mode is therefore characterized by the innermost (smallest) stationary mirror radius among confined field modes which share the same spheroidal harmonic index l.
- For confined equatorial $m = l$ modes, the stationary mirror radius decreases with increasing values of the spheroidal harmonic index l.

From these characteristics of the stationary mirror radius, one concludes that the asymptotic radius

$$r^*_{\rm m} \equiv r^{\rm stat}_{\rm m}\ (\bar{a} \to 1, l = m \to \infty) \tag{3}$$

provides the innermost location of the confining mirror. The physical significance of this critical mirror radius, $r^*_{\rm m}$, lies in the fact that this is the innermost (*smallest*) radius of the confining mirror which allows the extraction of rotational energy from spinning Kerr BHs. Below we shall explore the physical properties of this critical mirror radius.

Cardoso et al. [7] also provided an analytic treatment of the BH–mirror-field system in the *small* frequency regime $a\omega \ll 1$. Substituting the critical (marginal) superradiant frequency $\omega_{\rm c} = m\Omega_{\rm H}$ [see Eq. (1)] into Eq. (30) of [7], one finds that the stationary mirror radius $r^{\rm stat}_{\rm m}(\bar{a}, l, m)$ is given as a solution of the characteristic equation

$$J_{l+1/2}\big(m\Omega_{\rm H} r^{\rm stat}_{\rm m}\big) = 0, \tag{4}$$

where $J_{l+1/2}(x)$ is the Bessel function. It should be emphasized that the characteristic equation (4) for the 'stationary' (critical) radii of the mirror is valid in the regime $a\omega \ll 1$ considered in [7]. Thus, the characteristic equation (4) can only determine the stationary mirror radii of *slowly*-rotating BHs in the regime $m\bar{a} \ll 1$.

One of the goals of the present study is to obtain an analogous characteristic equation for the stationary mirror radii of *rapidly*-rotating $(\bar{a} \simeq 1)$ BHs. As discussed above, the *numerical* results presented in [7] indicate that these near-extremal BHs are characterized by stationary mirror radii which are *closer* to the BH horizon than the corresponding stationary mirror radii (4) of slowly-rotating BHs. Below we shall confirm this expectation *analytically*.

In addition, as we shall show below, our characteristic equation for the critical ('stationary' [6]) mirror radii of rapidly-rotating BHs [see Eq. (10) below] is valid for arbitrarily large values of the harmonic indexes (l, m) of the confined field [10]. This fact will allow us to address the following interesting question regarding the nature of this critical mirror radius: what is the *asymptotic* behavior of the stationary mirror radius in the eikonal $l \gg 1$ limit?

As discussed above, the asymptotic mirror radius, $r^*_{\rm m} \equiv r^{\rm stat}_{\rm m}(\bar{a} \to 1, l = m \to \infty)$, corresponds to the *innermost* location of the confining mirror which allows the extraction of rotational energy from spinning Kerr BHs. In addition, for generic confined field configurations [11], this critical mirror radius marks the boundary between stable and explosive BH–mirror-field configurations. One of the goals of the present study is to determine this fundamental (asymptotic) mirror radius $r^*_{\rm m}$.

2. Description of the system

The explored physical system is composed of a spinning Kerr black hole of mass M and angular momentum Ma linearly coupled to a massless scalar field Ψ. The radii of the BH (event and inner) horizons are given by: $r_\pm = M \pm (M^2 - a^2)^{1/2}$.

As discussed above, the critical ('stationary' [6]) mirror radius which characterizes the composed BH–mirror-field system is a decreasing function of the BH rotation parameter $\bar{a}$. Thus, rapidly-rotating black holes are expected to be characterized by the smallest (*innermost*) stationary mirror radii. In the present study we shall analyze the physical properties of the BH–mirror-field system in this physically interesting regime of rapidly-rotating Kerr BHs with

$$\bar{a} \simeq 1. \tag{5}$$

The dynamics of the scalar field Ψ in the curved geometry is determined by the Klein–Gordon wave equation $\nabla^a \nabla_a \Psi = 0$. It is convenient to decompose the scalar field Ψ in the form [12,13] $\Psi = \sum_{l,m} e^{im\phi} S_{lm}(\theta; a\omega) R_{lm}(r; a, \omega) e^{-i\omega t}$. The angular functions $S_{lm}(\theta; a\omega)$ are known as the spheroidal harmonic functions [14,15]. Regular angular eigenfunctions exist for a discrete set $\{A_{lm}(a\omega)\}$ of angular eigenvalues which are labeled by the two integers m and $l \geq |m|$ [16,17]. Following [7,9] we shall assume that the radial eigenfunction vanishes at the location $r_{\rm m}$ of the confining mirror:

$$R(r = r_{\rm m}) = 0. \tag{6}$$

3. Marginally stable black-hole–mirror-field configurations

We shall now analyze the *stationary* $(\Im\omega = 0)$ resonances of the composed system. As discussed above, these stationary (marginally-stable) BH–mirror-field configurations are characterized by the critical frequency (1) for superradiant scattering in the BH spacetime. In particular, in this section we shall find the *discrete* set of mirror radii, $\{r^{\rm stat}_{\rm m}(\bar{a}, l, m; n)\}$, which support stationary confined field configurations. (Here $n = 1, 2, 3, \dots$ is the resonance parameter.)

We consider a rapidly-rotating Kerr BH [18] surrounded by a reflecting mirror which is placed in the vicinity of the BH horizon. In particular, we shall assume the following inequalities:

$$\tau \ll 1 \quad \text{and} \quad x_{\rm m} \ll 1, \tag{7}$$

where

$$\tau \equiv 8\pi M T_{\rm BH} = \frac{r_+ - r_-}{r_+}; \qquad x \equiv \frac{r - r_+}{r_+}. \tag{8}$$

The physical [19] radial solution in the near-horizon region $x \ll 1$ corresponding to the critical (marginally-stable) superradiant frequency (1) is given by [14,17] $R(x) = (\frac{x}{\tau} + 1)^{-ik}\,{}_2F_1(\frac{1}{2} - ik +$

Table 1
Stationary resonances of the composed BH–mirror-field system. We display the dimensionless mirror radii, $x_{\text{m}}^{\text{stat}}(m;n)/\tau$, corresponding to stationary confined modes. The data presented is for the equatorial $l=m$ modes, the modes with the smallest (innermost) stationary mirror radii. Also shown are the corresponding values of the angular eigenvalues δ [see Eq. (9)]. One finds that the critical (stationary) radius of the mirror, $x_{\text{m}}^{\text{stat}}(m;n)$, is a decreasing function of the azimuthal harmonic index m.

$l=m$	δ	$x_{\text{m}}^{\text{stat}}(n=1)/\tau$	$x_{\text{m}}^{\text{stat}}(n=2)/\tau$	$x_{\text{m}}^{\text{stat}}(n=3)/\tau$	$x_{\text{m}}^{\text{stat}}(n=4)/\tau$
2	0.945	91.79	2610.95	72 601.28	2 017 165.8
3	1.937	10.99	62.43	322.69	1640.27
4	2.849	5.23	18.85	59.77	183.00
5	3.739	3.45	9.95	24.93	59.59

Table 2
Stationary resonances of the composed BH–mirror-field system. We display the dimensionless mirror radii, $x_{\text{m}}^{\text{stat}}(m;n=1)/\tau$, corresponding to stationary confined modes in the asymptotic $l=m\gg 1$ regime. One finds that the critical (stationary) radius of the mirror, $x_{\text{m}}^{\text{stat}}(m;n=1)$, decreases monotonically to an asymptotic *finite* value [see Eq. (16)] in the eikonal $l\gg 1$ limit.

$l=m$	25	50	75	100	125	150	175	200
$x_{\text{m}}^{\text{stat}}(m;n=1)/\tau$	0.75	0.55	0.49	0.46	0.44	0.43	0.42	0.41

$i\delta, \frac{1}{2}-ik-i\delta;1;-x/\tau)$, where ${}_2F_1(a,b;c;z)$ is the hypergeometric function [17], and

$$\delta^2\equiv -a^2\omega^2+2ma\omega-A+k^2-\frac{1}{4};\qquad k\equiv 2\omega r_+. \tag{9}$$

We shall henceforth consider the case of real δ [20,21]. The mirror-like boundary condition $R(x=x_{\text{m}}^{\text{stat}})=0$ [see Eq. (6)] now reads

$${}_2F_1\big(1/2-ik+i\delta,1/2-ik-i\delta;1;-x_{\text{m}}^{\text{stat}}/\tau\big)=0. \tag{10}$$

It is worth emphasizing that the newly derived resonance condition (10) for the critical ('stationary' [6]) mirror radii of the system is valid for confining mirrors which are placed in the near-horizon region $x_{\text{m}}\ll 1$. [Below we shall see that this near-horizon condition implies that the stationary mirror radii obtained from (10) are valid in the regime of *rapidly*-rotating BHs with $\tau\ll 1$]. On the other hand, the resonance condition (4) [7] for the stationary mirror radii is valid in the complementary regime $x_{\text{m}}\gg 1$, or equivalently in the regime of *slowly*-rotating BHs with $m\bar{a}\ll 1$.

One important conclusion which can immediately be drawn from the resonance condition (10) is the fact that, for rapidly-rotating BHs, the critical mirror radius scales linearly with the BH temperature [22]:

$$x_{\text{m}}^{\text{stat}}\propto\tau. \tag{11}$$

This implies that the stationary mirror radius $x_{\text{m}}^{\text{stat}}$ is a decreasing function of the BH rotation parameter $\bar{a}$ (an increasing function of the BH dimensionless temperature τ).

For small and moderate values of the field harmonic indexes (l,m), the stationary mirror radii $x_{\text{m}}^{\text{stat}}(\bar{a},l,m;n)$ are characterized by the relation (see Table 1)

$$x_{\text{m}}^{\text{stat}}/\tau\gg 1, \tag{12}$$

in which case the resonance condition (10) can be solved *analytically*. In the regime (12) one can use the large-z asymptotic behavior of the hypergeometric function ${}_2F_1(a,b;c;z)$ [17] to approximate the resonance condition (10) by

$$\frac{\Gamma(2i\delta)}{\Gamma(1/2-ik+i\delta)\Gamma(1/2+ik+i\delta)}\bigg(\frac{x_{\text{m}}^{\text{stat}}}{\tau}\bigg)^{i\delta}+\frac{\Gamma(-2i\delta)}{\Gamma(1/2-ik-i\delta)\Gamma(1/2+ik-i\delta)}\bigg(\frac{x_{\text{m}}^{\text{stat}}}{\tau}\bigg)^{-i\delta}=0. \tag{13}$$

This yields [23]

$$x_{\text{m}}^{\text{stat}}(n)=\tau\times\bigg[\frac{\Gamma(-2i\delta)\Gamma(1/2+ik+i\delta)\Gamma(1/2-ik+i\delta)}{\Gamma(2i\delta)\Gamma(1/2-ik-i\delta)\Gamma(1/2+ik-i\delta)}\bigg]^{1/2i\delta}\times e^{\pi(2n-1)/2\delta} \tag{14}$$

for the discrete set $\{x_{\text{m}}^{\text{stat}}(\bar{a},l,m;n)\}$ of stationary [6] mirror radii, where $n=1,2,3,\ldots$ is the resonance parameter of the mode.

In Table 1 we display the discrete set of mirror radii, $\{x_{\text{m}}^{\text{stat}}(l=m;n)/\tau\}$, corresponding to composed BH–mirror configurations with stationary confined equatorial $(l=m)$ field modes [24,25]. The data presented in Table 1 correspond to a direct numerical solution [26] of the characteristic resonance condition (10). It is worth recalling that, for a confined pure field (that is, a confined mode characterized by a given value of the azimuthal harmonic index m), the smallest stationary mirror radius, $x_{\text{m}}^{\text{stat}}(\bar{a},m;n=1)$, marks the boundary between stable and unstable BH–mirror-field configurations for that particular pure mode.

From Table 1 one learns that the stationary mirror radius, $x_{\text{m}}^{\text{stat}}(\bar{a},m;n=1)$, is a *decreasing* function of the azimuthal harmonic index m. This is a generic feature of the composed BH–mirror-field system: in Table 2 we display the values of the stationary mirror radii in the asymptotic $m\gg 1$ regime [27,28]. One finds that the data presented in Table 2 is described extremely well by the simple asymptotic formula:

$$\frac{x_{\text{m}}^{\text{stat}}(\bar{a}\to 1,l=m\gg 1)}{\tau}\simeq\alpha+\frac{\beta}{m}+O\big(m^{-2}\big)\quad\text{with }\alpha\simeq 0.36;\ \beta\simeq 10.5. \tag{15}$$

What we find most interesting is the fact that the coefficient α in (15) has a finite asymptotic value. This fact suggests that the composed Kerr-BH–mirror-field system is characterized by a *finite* asymptotic limit of the critical ('stationary') mirror radius:

$$\frac{x_{\text{m}}^*}{\tau}\simeq 0.36, \tag{16}$$

where $x_{\text{m}}^*\equiv x_{\text{m}}^{\text{stat}}(\bar{a}\to 1,l=m\to\infty)$ [see Eq. (3)].

In order to support our findings, according to which the composed Kerr-BH–mirror-field system is characterized by a *finite* asymptotic value of the critical mirror radius, we shall prove in the next section that composed systems whose mirror radii lie in the regime $x_{\text{m}}/\tau\ll 1$ *cannot* support stationary confined field configurations.

4. No stationary confined field configurations in the regime $x_{\text{m}}/\tau\ll 1$

We have seen that the (scaled) critical mirror radius, $x_{\text{m}}^{\text{stat}}(m)/\tau$, decreases monotonically with increasing values of the azimuthal harmonic index m. This fact naturally gives rise to the following question: Can the ratio $x_{\text{m}}^{\text{stat}}(m)/\tau$ be made arbitrarily small in the asymptotic eikonal regime $m\to\infty$? One can also formulate this question in a more practical form: Is it possible to extract the BH rotational energy by placing a confining mirror arbitrarily close to its horizon [29]?

Our analysis in Section 3 provides compelling evidence that the answer to the above questions is "no". In particular, our results suggest that the Kerr-BH–mirror-field system is characterized by a *finite* asymptotic value of the critical ('stationary' [6]) mirror radius: $x_{\mathrm{m}}^{\mathrm{stat}}(m \gg 1)/\tau = O(1)$ [see Eq. (16)]. In order to support this finding, we shall prove in this section that composed BH–mirror systems whose mirror radii lie in the regime $x_{\mathrm{m}}/\tau \ll 1$ cannot support stationary confined field configurations [30].

To that end, it proves useful to write the radial Teukolsky equation in the form of a Schrödinger-like wave equation [31]:

$$\frac{d^2\psi}{dy^2} + V\psi = 0, \tag{17}$$

where $\psi = (r^2+a^2)^{1/2}R$ and the "tortoise" radial coordinate y is defined by $dy = [(r^2+a^2)/\Delta]dr$. The potential V in Eq. (17) is given by [31]

$$V = \frac{K^2 - \Delta\lambda}{(r^2+a^2)^2} - G^2 - \frac{dG}{dy}, \tag{18}$$

where $G \equiv r\Delta/(r^2+a^2)^2$ and $\lambda \equiv k^2 - \frac{1}{4} - \delta^2$. In the eikonal regime, $l, m \gg 1$, one can use the relation [28]

$$\delta = l \times \sqrt{-1 + \frac{15}{8}\mu^2 - \frac{1}{8}\mu^4} + O(1); \quad \mu \equiv \frac{m}{l}, \tag{19}$$

in order to find $\lambda = l^2(1 - \frac{7}{8}\mu^2 + \frac{1}{8}\mu^4)$. One therefore concludes that

$$\lambda > 0 \tag{20}$$

for all values of the dimensionless ratio μ (note that $\mu \le 1$).

In the near-horizon $x \ll \tau$ region one finds

$$y \simeq \frac{r_+^2 + a^2}{r_+ - r_-} \ln\left(\frac{r - r_+}{r_+ - r_-}\right) = \frac{2M}{\tau} \ln\left(\frac{x}{\tau}\right), \tag{21}$$

which implies $\Delta \simeq (r_+ - r_-)^2 e^{\tau y/2M}$ and

$$V(y) \simeq (\omega - m\Omega_{\mathrm{H}})^2 - V_{\mathrm{H}} e^{\tau y/2M};$$
$$V_{\mathrm{H}} \equiv \left(\frac{\tau}{2M}\right)^2 \left(\lambda + \frac{\tau r_+}{2M}\right). \tag{22}$$

From Eqs. (17) and (22) one obtains the radial wave equation

$$\frac{d^2\psi}{dy^2} - V_{\mathrm{H}} e^{\tau y/2M} \psi = 0 \tag{23}$$

for stationary field configurations (with $\omega = m\Omega_{\mathrm{H}}$). Using Eq. (9.1.54) of [17], one finds that the physical solution to the radial equation (23) is described by the Bessel function of the first kind:

$$\psi(x) = J_0\left(2i\sqrt{\left(\lambda + \frac{\tau r_+}{2M}\right)\frac{x}{\tau}}\right). \tag{24}$$

Taking cognizance of Eq. (20), one finds that the argument of the Bessel function in (24) is purely *imaginary*. It is well known that the Bessel function $J_0(ix)$ with $x \in \mathbb{R}$ has no zeroes [that is, $\psi(x_{\mathrm{m}}) \neq 0$ for $x_{\mathrm{m}} \in \mathbb{R}$]. One therefore concludes that stationary solutions (with $\omega = m\Omega_{\mathrm{H}}$) of the wave field are *not* compatible with the mirror-like boundary condition (6) for confining mirrors in the region $x_{\mathrm{m}} \ll \tau$.

The proof presented in this section supports our previous conclusion that, the composed Kerr-BH–mirror-field system is characterized by a *finite* asymptotic value [see Eq. (16)] of the dimensionless ratio x_{m}^*/τ, where x_{m}^* is the critical mirror radius.

5. Summary and discussion

In this paper, we have used analytical tools in order to study the *stationary* (marginally-stable) resonances of the composed BH–mirror-field system. These resonances are fundamental to the physics of confined bosonic fields in BH spacetimes. In particular, these resonances mark the onset of superradiant instabilities in the BH bomb mechanism of Press and Teukolsky [4].

We have derived the characteristic resonance condition (10) for the marginally-stable (stationary) BH–mirror-field configurations. In particular, it was shown that, for rapidly-rotating BHs, the stationary resonances of the system are described by the simple zeroes of the hypergeometric function.

The characteristic resonance condition (10) determines the discrete set of mirror radii, $\{x_{\mathrm{m}}^{\mathrm{stat}}(\bar{a}, l, m; n)\}$, which support stationary confined field configurations in the BH spacetime. One nice feature of this resonance condition lies in the fact that it immediately reveals that the critical ('stationary' [6]) mirror radii scale linearly with the BH temperature [22], see Eq. (11). This fact implies that $x_{\mathrm{m}}^{\mathrm{stat}}(\bar{a}, l, m; n)$ is a decreasing function of the BH rotation parameter $\bar{a}$ — the larger the BH spin, the closer to the BH horizon the confining mirror can be placed.

It was shown that the stationary mirror radius, $x_{\mathrm{m}}^{\mathrm{stat}}(m)$, decreases monotonically with increasing values of the azimuthal harmonic index m of the confined field mode. In particular, our results provide compelling evidence that the composed Kerr-BH–mirror-field system is characterized by a *finite* asymptotic value of the critical mirror radius: $x_{\mathrm{m}}^{\mathrm{stat}}(m \gg 1)/\tau \simeq 0.36$ [32].

The physical significance of the asymptotic stationary mirror radius, $x_{\mathrm{m}}^* \equiv x_{\mathrm{m}}^{\mathrm{stat}}(\bar{a} \to 1, l = m \to \infty)$, lies in the fact that it is the *innermost* location of the confining mirror which allows the extraction of rotational energy from spinning BHs. This implies that, for generic confined field configurations [11], this critical mirror radius marks the onset of superradiant instabilities in the composed system: composed BH–mirror-field systems whose mirror radii lie in the regime $x_{\mathrm{m}} < x_{\mathrm{m}}^*$ are stable (that is, all modes of the confined field decay in time), whereas composed BH–mirror-field systems whose mirror radii lie in the regime $x_{\mathrm{m}} > x_{\mathrm{m}}^*$ are unstable (that is, there are confined field modes which grow exponentially over time).

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[6] We use the terminology 'stationary mirror radius' in order to reflect the fact that this critical radius of the confining mirror supports stationary BH–field configurations with $\Im\omega = 0$. It is worth emphasizing again that, for a confined field mode of given harmonic indexes (l, m), the stationary mirror radius $r_{\mathrm{m}}^{\mathrm{stat}}(\bar{a}, l, m)$ marks the boundary between stable and unstable BH–field configurations. Thus, this critical mirror radius signals the onset of the superradiant instabilities in the composed BH–mirror-field system.

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[19] We are interested in radial solutions with the physical requirement (boundary condition) of purely ingoing waves (as measured by a comoving observer) crossing the BH horizon [14].
[20] Below we shall see that equatorial field modes with $l = m \geq 2$, the modes we shall be interested in this study, are characterized by this property.
[21] One can choose $\delta > 0$ without loss of generality.
[22] Note that the resonance condition (10) for the stationary confined states is expressed in terms of the dimensionless ratio $x^{\mathrm{stat}}_{\mathrm{m}}/\tau$.
[23] Here we have used the relation $-1 = e^{i\pi(2n-1)}$ which implies $\ln(-1) = i\pi(2n-1)$, where n is an integer.
[24] One of our goals is to determine the smallest mirror radius r^*_{m} [see Eq. (3)] which allows the extraction of rotational energy from spinning Kerr BHs. (This critical mirror radius also marks the boundary between stable and unstable BH–mirror-field configurations.) As discussed above, the numerical results of [7] indicate that, for a given value of the spheroidal harmonic index l of the confined field, the equatorial $m = l$ mode is characterized by the innermost location of the stationary mirror radius. We therefore focus our attention on these equatorial modes.
[25] It is worth emphasizing again that the mirror-like boundary condition (10) is valid for mirror radii in the $x_{\mathrm{m}} \ll 1$ regime [see Eq. (7)]. Solving the resonance condition (10) for small and moderate values of the spheroidal harmonic index l, one finds $x^{\mathrm{stat}}_{\mathrm{m}}(\bar{a}, m; n = 1)/\tau \gtrsim 1$, see Table 1. Thus, the stationary mirror radii presented in Table 1 are valid in the regime of rapidly-rotating (near-extremal) BHs with $\tau \lesssim x^{\mathrm{stat}}_{\mathrm{m}}(\bar{a}, m; n = 1) \ll 1$.
[26] It is worth emphasizing that, finding the roots of the characteristic equation (10) is a much easier task than the Runge–Kutta numerical solution of the Teukolsky wave equation presented in [7]. In addition, as we shall discuss below, another notable advantageous of our analysis lies in the fact that, the characteristic resonance condition (10) can easily be solved for asymptotically large values of the harmonic indexes (l, m). Below we shall be interested in this asymptotic (eikonal) regime.
[27] In the asymptotic $m \gg 1$ regime, the angular separation constants δ are given by Eq. (19) below, see [28].
[28] S. Hod, Phys. Lett. B 715 (2012) 348, arXiv:1207.5282.
[29] The answer to this question may one day be of practical importance: In order to save construction materials, one would like to build the confining mirror as close as possible to the BH horizon.
[30] The hypergeometric function ${}_2F_1(a, b; c; z)$ is characterized by the property ${}_2F_1(a, b; c; z) \to 1$ for $a \cdot b \cdot z/c \ll 1$ [17]. Taking cognizance of the characteristic resonance condition (10), one realizes that there are no confined stationary field configurations in the regime $m^2 x_{\mathrm{m}}/\tau \ll 1$ [in this regime one finds ${}_2F_1 \simeq 1$, whereas the resonance condition (10) for confined stationary states requires ${}_2F_1 \simeq 0$]. In this section we would like to prove the (stronger) claim that there are no stationary confined field configurations for mirror radii in the regime $x_{\mathrm{m}}/\tau \ll 1$.
[31] W.H. Press, S.A. Teukolsky, Astrophys. J. 185 (1973) 649.
[32] It is worth emphasizing that this feature of the *spinning* BH–mirror-field bomb is a non-trivial one. In particular, it should be contrasted with the *charged* BH–mirror-field bomb [9]. It was previously shown [9] that, in the charged case, the stationary mirror radius, $x^{\mathrm{stat}}_{\mathrm{m}}(qQ)$, can be made arbitrarily small in the asymptotic $qQ \gg 1$ regime (that is, in the charged case, the confining mirror can be placed arbitrarily close to the BH horizon [9]): $x^{\mathrm{stat}}_{\mathrm{m}}(qQ \gg 1)/\tau = O(1/qQ) \to 0$. [Here Q and q are respectively the electric charge of the Reissner–Nordström BH and the charge coupling constant of the confined field.]